Gradient in orthogonal curvilinear coordinates: –

Let $\nabla f = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3$, $f \rightarrow$ scalar function
$\hat{e} \rightarrow$ unit vectors
$f_1, f_2, f_3 \rightarrow$ are to be determined

$$\because d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$= h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3$$

We can write.

$$df = \nabla f \cdot d\vec{r} = h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3 \quad \text{——— (a)}$$

$$\text{and } df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3 \quad \text{——— (b)}$$

From (a) & (b) we can write

$$f_1 = \frac{1}{h_1}\frac{\partial f}{\partial u_1}, \quad f_2 = \frac{1}{h_2}\frac{\partial f}{\partial u_2}, \quad f_3 = \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\text{Thus } \nabla f = \frac{\hat{e}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial f}{\partial u_3}$$

$$\text{or } \boxed{\nabla \equiv \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}} \quad \text{——— (c)}$$

Divergence in orthogonal curvilinear coordinates

$$\text{from eq}^n \text{ (c)} \quad \nabla u_1 = \frac{\hat{e}_1}{h_1}, \quad \nabla u_2 = \frac{\hat{e}_2}{h_2}, \quad \nabla u_3 = \frac{\hat{e}_3}{h_3}$$

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{e}_2 \times \hat{e}_3}{h_2 h_3} = \frac{\hat{e}_1}{h_2 h_3} \Rightarrow \hat{e}_1 = h_2 h_3 (\nabla u_2 \times \nabla u_3)$$

Similarly $\hat{e}_2 = h_1 h_3 (\nabla u_1 \times \nabla u_3)$ & $\cancel{\hat{e}_3 = h_1 h_2 (\nabla u_1 \times \nabla u_2)}$

$$\hat{e}_3 = h_1 h_2 (\nabla u_1 \times \nabla u_2)$$

For a vector (field) $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$

$$\nabla\cdot\vec{A} = \nabla\cdot(A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) = \nabla\cdot(A_1\hat{e}_1) + \nabla\cdot(A_2\hat{e}_2) + \nabla\cdot(A_3\hat{e}_3) \quad \text{--- (d)}$$

Let us take $\nabla\cdot(A_1\hat{e}_1) = \nabla\cdot(A_1 h_2 h_3 \nabla u_2 \times \nabla u_3)$

or $\nabla\cdot(A_1\hat{e}_1) = \nabla(A_1 h_2 h_3)\cdot\nabla u_2\times\nabla u_3 + A_1 h_2 h_3 \nabla\cdot(\nabla u_2\times\nabla u_3)$

$$= \nabla(A_1 h_2 h_3)\cdot\frac{\hat{e}_2}{h_2}\times\frac{\hat{e}_3}{h_3} + 0 \quad \{\because \text{div(curl)} = 0\}$$

$$= \nabla(A_1 h_2 h_3)\cdot\frac{\hat{e}_1}{h_2 h_3}$$

Now from eqn (d)

$$\nabla\cdot(A_1\hat{e}_1) = \left[\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2}(A_1 h_2 h_3) + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}(A_1 h_2 h_3)\right]\cdot\frac{\hat{e}_1}{h_2 h_3}$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(A_1 h_2 h_3) \quad \{\because \hat{e}_1\cdot\hat{e}_2 = 0 = \hat{e}_3\cdot\hat{e}_1\}$$

Similarly $\nabla\cdot(A_2\hat{e}_2) = \dfrac{1}{h_1 h_2 h_3}\dfrac{\partial}{\partial u_2}(A_2 h_1 h_3)$

& $\nabla\cdot(A_3\hat{e}_3) = \dfrac{1}{h_1 h_2 h_3}\dfrac{\partial}{\partial u_3}(A_3 h_1 h_2)$

Now from (d)

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]}$$